801. Completing the square, we get

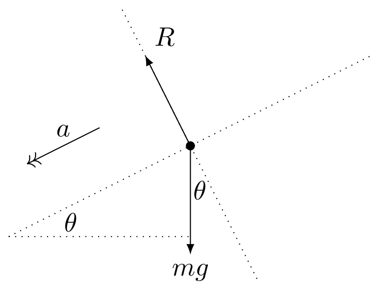
$$\mathrm{h}(x) = (8x^2 - 3)^2 - 4.$$

Since the square term has a minimum value of zero, the range of the function is $\{y \in \mathbb{R} : y \geq -4\}$ or equivalently $[-4, \infty)$.

802. Expanding binomially,

$$\begin{aligned}&\left(\sqrt{2}+\sqrt{3}\right)^4\\ &= \sqrt{2}^4 + 4\sqrt{2}^3\sqrt{3} + 6\sqrt{2}^2\sqrt{3}^2 + 4\sqrt{2}\sqrt{3}^3 + \sqrt{3}^4\\ &= 4 + 8\sqrt{6} + 36 + 12\sqrt{6} + 9\\ &= 49 + 20\sqrt{6}, \text{ as required.}\end{aligned}$$

803. The particle has forces as follows:



Resolving down the slope, NII is

$$\begin{aligned}mg\sin\theta &= ma\\ \Longrightarrow a &= g\sin\theta.\end{aligned}$$

So, the acceleration on a slope of inclination θ is $g\sin\theta$ ms^{-2}, as required. QED.

804. The possibility space consists of the ${}^5\mathrm{C}_3 = 10$ equally likely ways of choosing the three integers.

(a) Without the 1, there are ${}^4\mathrm{C}_3$ successful choices. This gives

$$p = \frac{{}^4\mathrm{C}_3}{{}^5\mathrm{C}_3} = \frac{2}{5}.$$

(b) To pick all three odd numbers, there is only one successful outcome. So,

$$p = \frac{1}{{}^5\mathrm{C}_3} = \frac{1}{10}.$$

805. By an index law, $x\sqrt{x} \equiv x^{\frac{3}{2}}$. Differentiating,

$$\begin{aligned}\mathrm{f}(x) &= x^{\frac{3}{2}} + 3x\\ \Longrightarrow \mathrm{f}'(x) &= \tfrac{3}{2}x^{\frac{1}{2}} + 3.\end{aligned}$$

Evaluating this, $\mathrm{f}'(1) = \frac{9}{2}$. This is greater than zero, so the function is increasing at this point.

806. (a) Multiplying out,

$$a^2 + 6b^2 + 2ab\sqrt{6} = 58 - 12\sqrt{6}.$$

Equating the rational and irrational terms, $a^2 + 6b^2 = 58$ and $ab = -6$.

(b) Substituting $a = -\frac{6}{b}$ gives

$$\begin{aligned}&\frac{36}{b^2} + 6b^2 = 58\\ \Longrightarrow\ & 6b^4 - 58b^2 + 36 = 0\\ \Longrightarrow\ & b^2 = \frac{58 \pm \sqrt{58^2 - 4\cdot 6\cdot 36}}{2\cdot 6}\\ \Longrightarrow\ & b^2 = 9, \tfrac{2}{3}.\end{aligned}$$

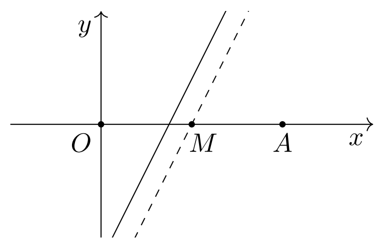
For $b \in \mathbb{Z}$, we want $b = \pm 3$. Hence, $a = \mp 2$. So, the square roots are $\pm 3 \mp 2\sqrt{6}$. Choosing the positive square root,

$$\sqrt{58 - 12\sqrt{6}} = 2\sqrt{6} - 3.$$

807. The midpoint of OA has coordinates $(2, 0)$. Since $0 < 2\cdot 2 - 3$, we know that $y < 2x - 3$ at this point. This is also true of $(4, 0)$. So, the midpoint of OA lies on the same side of the line as $(4, 0)$. Hence, the origin is closer.

— Alternative Method —

The line $y = 2x - 4$, dashed below, passes through the midpoint of OA, $M : (2, 0)$. By symmetry, this line is equidistant from O and A. Translating the line by vector $\mathbf{j}$ to $y = 2x - 3$ moves it closer to the origin.



Hence, O is closer.

808. The student's explanation is spot on. If the car were on an ice rink, then flooring the accelerator would result only in wheel spin. In other words, it is friction that stops the wheels spinning, and friction that drives the car forward.

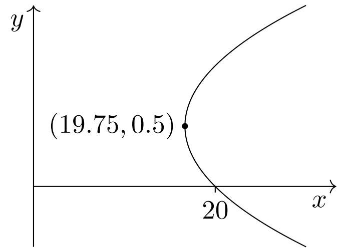
809. (a) Setting $y = 0$ gives the x intercept as $x = 20$. Setting $x = 0$ gives $y^2 - y + 20 = 0$. This has discriminant $\Delta = -79 < 0$. So, there are no y axis intercepts. The only intercept is $(20, 0)$.

(b) Differentiating with respect to y,

$$\begin{aligned}x &= y^2 - y + 20\\ \Longrightarrow \frac{dx}{dy} &= 2y - 1.\end{aligned}$$

When the tangent is parallel to y, $dx/dy = 0$, which occurs at $y = \frac{1}{2}$. Substituting gives the vertex as $(19.75, 0.5)$.

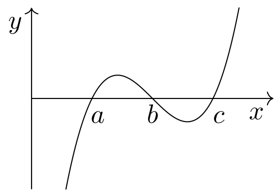
(c) This is a positive parabola. Its vertex is at the point $(19.75, 0.5)$. Not to scale, its graph is



810. Writing the sum longhand,

$$\begin{aligned}&\frac{1-x}{x}+\frac{2-x}{2x}+\frac{3-x}{3x}=\tfrac{1}{2}\\ \Longrightarrow\ &\frac{6(1-x)+3(2-x)+2(3-x)}{6x}=\tfrac{1}{2}\\ \Longrightarrow\ &18-11x=3x\\ \Longrightarrow\ &x=\tfrac{9}{7}.\end{aligned}$$

811. Consider the graph $y=(x-a)(x-b)(x-c)$. This is a positive cubic, with three distinct single roots. So, the behaviour is as shown:



We are looking for $y>0$. So, the solution to the inequality is $x\in(a,b)\cup(c,\infty)$.

— Alternative Method —

The expression $(x-a)(x-b)(x-c)$ is a positive cubic, with three distinct single roots. Hence, the sign of the cubic changes at each root. For large negative values of x, all three factors are negative, so their product is negative. Passing $x=a$, the sign changes to positive; passing $x=b$, it changes again to negative; passing $x=c$, it changes again to positive. So, the solution is $x\in(a,b)\cup(c,\infty)$.

812. The roles of x (originally inputs) and y (originally outputs) have been reversed in the transformed graph. This takes the point (p,q) to the point (q,p), which is reflection in the line $y=x$.

813. Consider the rolls one after another. The first can be anything, probability 1. For success, the second must differ from the first; this has probability $5/6$. Two numbers are now taken. For success, the third roll must differ from these; this has probability $4/6$. So, the probability is

$$p=1\times\tfrac{5}{6}\times\tfrac{4}{6}=\tfrac{5}{9}.$$

814. (a) A unit circle has radius 1. So, a perpendicular dropped from the point $(\cos\theta,\sin\theta)$ to the x axis produces a right-angled triangle with sides $(\sin\theta,\cos\theta,1)$. Pythagoras gives the required result: the first Pythagorean trig identity.

(b) Dividing the first identity by $\cos^2\theta$, we use the definitions $\tan\theta\equiv{}^{\sin\theta}/_{\cos\theta}$ and $\sec\theta\equiv{}^{1}/_{\cos\theta}$:

$$\begin{aligned}&\sin^2\theta+\cos^2\theta\equiv1\\ \Longrightarrow\ &\frac{\sin^2\theta}{\cos^2\theta}+1\equiv\frac{1}{\cos^2\theta}\\ \Longrightarrow\ &\tan^2\theta+1\equiv\sec^2\theta.\end{aligned}$$

This is the second Pythagorean trig identity.

815. The gradient of such a parametrically defined line is the ratio of the coefficients of the parameter. These are $\frac{4}{-2}=-2$ and $\frac{-2}{1}=-2$. The lines have the same gradient, so they are parallel.

816. The original sample has

$$\sum x=21.2\times100=2120.$$

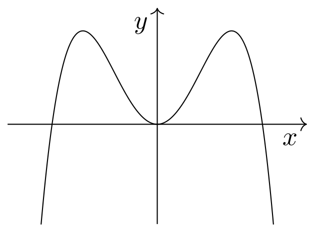
The set of 10 data has

$$\sum x=19.4\times10=194.$$

Subtracting, the remaining 90 have $\sum x=1926$. The raised mean is therefore

$$\bar{x}=\frac{1926}{90}=21.4.$$

817. Since this is a negative quartic, it has a maximum which occurs at a stationary point. Differentiating and setting to zero, we get $-80x^3+1600x=0$. This has three roots, at $x=0,\pm\sqrt{20}$. By the shape of a quartic, the maximum must be at $x=\pm\sqrt{20}$.



Substituting $x=\sqrt{20}$, we get $20^3=8000$.

— Alternative Method —

This is a quadratic in x^2. Multiplying out and completing the square,

$$\begin{aligned}&20x^2(40-x^2)\\ \equiv\ &-20x^4+800x^2\\ \equiv\ &8000-20(x^2-20)^2.\end{aligned}$$

The minimum value of the squared term is zero, so the greatest possible value is 8000.

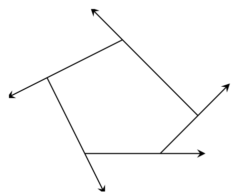
— Alternative Method —

The quadratic in x^2 has roots at $x^2 = 0, 40$. So, it is symmetrical around $x^2 = 20$. Substituting this value in,

$$\begin{aligned} & 20x^2(40 - x^2)\Big|_{x^2=20} \\ &= 20 \times 20 \times (40 - 20) \\ &= 8000, \text{ as required.} \end{aligned}$$

818. (a) $\text{gf}(a) = \text{g}(b) = d$,
(b) $\text{f}^{-1}\text{g}^{-1}(d) = \text{f}^{-1}(b) = a$,
(c) $\text{g}^{-1}\text{fg}(a) = \text{g}^{-1}\text{f}(c) = \text{g}^{-1}(d) = b$.

819. Consider a journey around the perimeter of the polygon:



At each vertex, we turn through an exterior angle. In one full journey around the perimeter, we turn through one full revolution, which is 2π radians by definition. Hence, the sum of the exterior angles of a polygon is 2π radians. □

820. (a) From the definition of the common difference d, we know that $u_{n+1} = u_n + d$. Multiplying this by q, $qu_{n+1} = qu_n + dq$, as required.
(b) Substituting the result of (a) into the formula for w_n,

$$\begin{aligned} w_n &= pu_n + qu_{n+1} \\ &= pu_n + qu_n + qd \\ &\equiv (p+q)u_n + qd. \end{aligned}$$

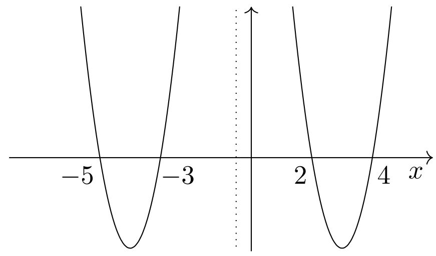
We have scaled u_n by $(p+q)$ and added qd. Both operations maintain the existence of a common difference. For u_n, this was d, so, for w_n, it is $d(p+q)$.

821. Horizontally, there is no velocity, so the distance remains constant. Vertically, both objects have the same free-fall acceleration and zero initial velocity. Dropped from heights h_1 and h_2, their positions at time t are given by

$$\begin{aligned} y_1 &= h_1 - \tfrac{1}{2}gt^2, \\ y_2 &= h_2 - \tfrac{1}{2}gt^2. \end{aligned}$$

The difference $y_1 - y_2$ between these is a constant $h_1 - h_2$, the difference between the initial heights. Since both horizontal and vertical distances are constant, the overall distance between the objects is constant. QED.

822. The first quadratic has roots at $x = 2$ and $x = 4$. Factorising, the second is $y = 3(x+3)(x+5)$, which has roots at $x = -3$ and $x = -5$. Since the quadratics have the same leading coefficient and their roots are the same distance apart, they are reflections in a line of the form $x = k$.



Taking the mean of the outermost roots $x = -5$ and $x = 4$, we see that $k = {}^1\!/\!_2(4 - 5) = -{}^1\!/\!_2$. So, the line of symmetry is $x = -{}^1\!/\!_2$.

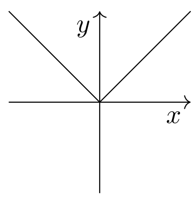
823. Let the squares have unit side length. The grid has area 9. From this, we remove a square (top left) of area 1, and four right-angled triangles, whose areas are, from top to bottom, $\{0.5, 1, 1, 1.5\}$. The shaded area is therefore 4. So, the fraction of the total area which is shaded is $\frac{4}{9}$.

824. The discriminant of the quadratic is $\Delta = -7 < 0$, so it has no real roots. And the factor theorem relates roots and factors. Hence, we cannot, while working in $\mathbb{R}$, use the factor theorem to establish whether it is a factor of the given quintic.

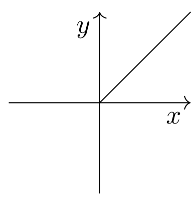
— Nota Bene —

It is possible to use the factor theorem in this case if one works with the *complex numbers* $\mathbb{C}$, which is the reason for the wording “over the real numbers” in the question. In the extended number system $\mathbb{C}$, negative numbers have square roots and the quadratic $2x^2 + x + 1$ is factorisable.

825. (a) The graph $y = \sqrt{x^2}$ is defined for all real x, and is the same as $y = |x|$.



(b) The graph $y = \left(\sqrt{x}\right)^2$ is only well defined for positive x, over which domain it is the same as $y = x$:



826. The first expansion is

$$\begin{aligned}&(2x+3)^4\\ \equiv\;& (2x)^4+4(2x)^3\cdot 3+6(2x)^2\cdot 3^2+4(2x)\cdot 3^3+3^4\\ \equiv\;& 16x^4+96x^3+216x^2+216x+81.\end{aligned}$$

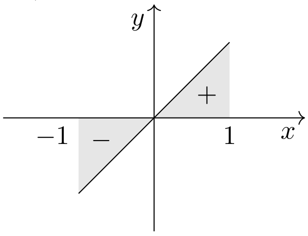
The second expansion is much the same, but it has alternating signs:

$$\begin{aligned}&(2x-3)^4\\ \equiv\;& 16x^4-96x^3+216x^2-216x+81.\end{aligned}$$

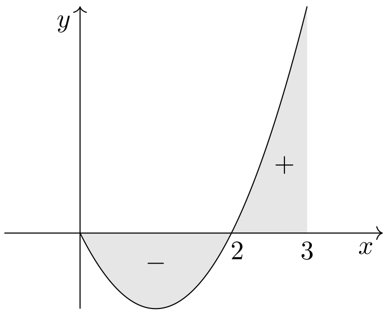
Adding these, the odd powers of x cancel, leaving only even powers. This gives

$$(2x+3)^4+(2x-3)^4\equiv 32x^4+432x^2+162.$$

827. (a) $\displaystyle\int_{-1}^{1} x\,dx=0,$



(b) $\displaystyle\int_{0}^{3} x^2-2x\,dx=0,$



828. Dividing both sides by $\cos\theta$,

$$\begin{aligned}&\sin x=-\sqrt{3}\cos x\\ \Longrightarrow\;& \frac{\sin x}{\cos x}=-\sqrt{3}\\ \Longrightarrow\;& \tan x=-\sqrt{3}\\ \Longrightarrow\;& x=-60°,120°.\end{aligned}$$

829. Separating the variables,

$$\begin{aligned}&\frac{dy}{dx}-4xy=0\\ \Longrightarrow\;& \frac{dy}{dx}=4xy\\ \Longrightarrow\;& \frac{1}{y}\frac{dy}{dx}=4x.\end{aligned}$$

——— Nota Bene ———

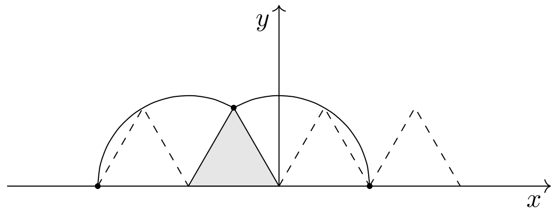
Dividing by $4y$ would give you the perfectly correct answer

$$\frac{1}{4y}\frac{dy}{dx}=x.$$

However, for future reference, it is easier to leave constants in the numerators of fractions than in their denominators. The original solution will be easier to deal with in later techniques.

830. Since the angles are in AP, the middle angle must be the average of the three. The total is 180°, so the average is 60°. Subtracting this from 180°, the sum of the other two is 120° (or $\frac{2\pi}{3}$ radians).

831. Each of the sections of the path of the vertex is a circular arc of radius 1:



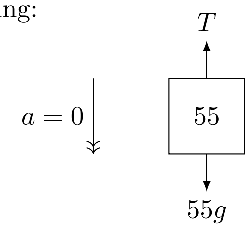
After these first two, the third rotation leaves the vertex where it is. This is one full cycle. So, the two circular arcs depicted are a single period of the cyclogon. Together, they form $\frac{2}{3}$ of a circle, with arc length $l=\frac{2}{3}\cdot 2\pi=\frac{4\pi}{3}$.

832. Assume, for a contradiction, that three distinct points on a parabola $y=ax^2+bx+c$ are collinear. Let this line be $y=px+q$. For intersections of the parabola and the line, the equation, which has three distinct roots, is

$$ax^2+bx+c=px+q.$$

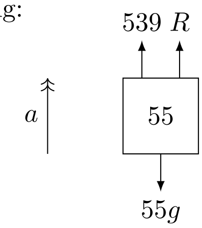
But this is a quadratic equation with at most two roots. This is a contradiction. Hence, no three distinct points on a parabola are collinear. □

833. (a) Before landing:



(b) At this stage, $T-55g=0$, so $T=539N$.

(c) During landing:

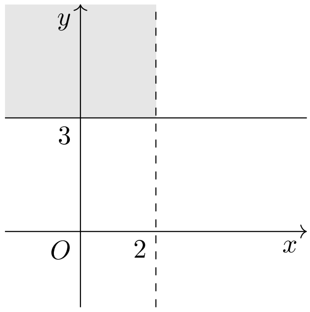


(d) During the landing, the average acceleration is $\frac{1.5}{0.4} = 3.75$ ms^{-2}. So, NII is

$$R + 55g - 55g = 55 \cdot 3.75,$$

which gives an average force of 206.25 N.

834. The boundary equations are lines $x = 2$ and $y = 3$. We require the region to the left of $x = 2$, and above and including $y = 3$. For this, we draw the line $y = 3$ as solid, and the line $x = 2$ as dashed:



835. The odd and even cases are different. Since the square renders all numbers positive, the second function may be defined over $\mathbb{R}$, unlike the other two.

(a) $[0, \infty)$,

(b) $\mathbb{R}$,

(c) $[0, \infty)$.

836. Translating the facts into algebra, $\text{f}'(x) = a$ and $g'(x) = b$, for constants $a, b \in \mathbb{R}$. Integrating gives $\text{f}(x) = ax + c$ and $\text{g}(x) = bx + d$. So, f and g are two linear functions. The equation $\text{f}(x) = \text{g}(x)$ is linear, therefore. It can have

- no roots, if $a = b$ and $c \neq d$,
- 1 root, if $a \neq b$, or
- infinitely many roots, if $a = b$ and $c = d$.

837. Parabola P_2 has been translated by $k\mathbf{i}$, so it has equation

$$y = (x - k)^2.$$

At $y = 16$, parabola P_1 has coordinates $(\pm 4, 16)$, so the curves intersect at one of these two points. Hence, we know that $16 = (\pm 4 - k)^2$. Taking the square root gives $\pm 4 = \pm 4 - k$, where the two $\pm$ signs are independent of one another. Considering all combinations, $k \in \{-8, 0, 8\}$.

— Nota Bene —

The value $k = 0$ could reasonably be excluded from the solution, as it doesn't represent a translation as such. However, it is logically valid: if there is no translation, then the parabolae P_1 and P_2 are identical, so they do intersect at $y = 16$. So, a good solution should consider $k = 0$, if only to reject it.

838. Multiplying out and differentiating,

$$\begin{aligned}&\frac{d}{dx}\Big(\tfrac{1}{2}x^2\big(\tfrac{1}{3}x^3 + x\big) + c\Big)\\ \equiv\ &\frac{d}{dx}\big(\tfrac{1}{6}x^5 + \tfrac{1}{2}x^3 + c\big)\\ \equiv\ &\tfrac{5}{6}x^4 + \tfrac{3}{2}x^2.\end{aligned}$$

This is not the original integrand: it is quartic not cubic. So, the student's claim is incorrect: you cannot integrate factor by factor.

839. The number of grains of rice is a GP with first term 1, common ratio 2, and number of terms 64. The total is therefore

$$N = \frac{1(2^{64} - 1)}{2 - 1} \approx 2^{64}.$$

One grain of rice is 10^{-2} grams, which is 10^{-5} kg, which is 10^{-8} tonnes. So, the total mass demanded is approximately $2^{64} \times 10^{-8}$, which is 1.84×10^{11}, or 184 billion tonnes.

840. Multiplying up by the denominators,

$$\begin{aligned}&\frac{1}{1 + \sqrt{x}} + \frac{1}{1 - \sqrt{x}} = 1\\ \Longrightarrow\ &1 - \sqrt{x} + 1 + \sqrt{x} = 1 - x\\ \Longrightarrow\ &x = -1.\end{aligned}$$

But the square root function $x \mapsto \sqrt{x}$ is undefined for negative x. So, $x = -1$ does not satisfy the equation, which therefore has no roots.

— Nota Bene —

Such problems are the reason for using implication arrows in algebra. A basic interpretation of the logic is: the equation "goes to" the value $x = -1$, so why shouldn't $x = -1$ be a root?

The correct meaning of the implication $A \Longrightarrow B$ is "If A then B". Depending on context, you may or may not *also* be saying that A is definitely true. In the context of this problem, this means that, *if* the original equation were to be satisfied, then nothing but $x = -1$ could satisfy it. This doesn't mean that $x = -1$ satisfies it.

841. The possibility space is contained in a 5×5 grid. Since the integers are distinct, however, the shaded outcomes on the leading diagonal are excluded.

	1	2	3	4	5
1		✓	✓	✓	✓
2	✓		✓	✓	
3	✓	✓			
4	✓	✓			
5	✓				

The outcomes are equally likely, so

$$\mathbb{P}(x_1 x_2 < 10) = \tfrac{\text{successful}}{\text{total}} = \tfrac{12}{20} = \tfrac{3}{5}.$$

842. (a) i. The gradient of the hypotenuse is $-\frac{b}{a}$. The point $(0, b)$ gives $y = b - \frac{b}{a}x$.

ii. Radius and hypotenuse are perpendicular. So, the gradient of the radius is $\frac{a}{b}$. It passes through the origin, so has equation $y = \frac{a}{b}x$.

(b) Solving the above equations simultaneously,

$$\begin{aligned} b - \frac{b}{a}x &= \frac{a}{b}x \\ \Longrightarrow ab^2 &= \left(a^2 + b^2\right)x \\ \Longrightarrow x &= \frac{ab^2}{a^2 + b^2} \\ &\equiv \frac{ab^2}{c^2}. \end{aligned}$$

By symmetry, the y coordinate is the same with a and b switched. So, point P is at

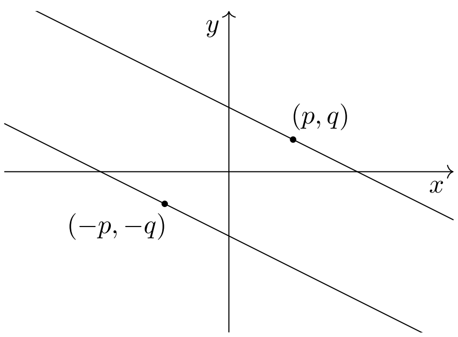
$$\left(\frac{ab^2}{c^2}, \frac{a^2 b}{c^2}\right).$$

(c) The radius is $|OP|$. So, using Pythagoras on the coordinates from part (b), the area of the circle is

$$\begin{aligned} A_{\text{circle}} &= \pi \frac{a^2 b^4 + a^4 b^2}{c^4} \\ &\equiv \frac{\pi a^2 b^2 \left(a^2 + b^2\right)}{c^4} \\ &\equiv \frac{\pi a^2 b^2}{c^2}, \text{ as required.} \end{aligned}$$

843. This is a well defined function. The discriminant of $x^2 + x + 1$ is $\Delta = -3 < 0$, so the denominator cannot equal zero. Hence, $\mathrm{g}(x)$ is well-defined for all $x \in \mathbb{R}$.

844. These are two straight lines, with negative gradient k, passing through the points (p, q) and $(-p, -q)$:



845. Since $1 + x^2$ is an irreducible quadratic (no roots) over the reals, we must factorise explicitly:

$$1 + x + x^2 + x^3 \equiv (1 + x^2)(1 + x).$$

846. The shaded region is a rhombus consisting of two equilateral triangles. The height of the hexagon is $\sqrt{3}$, and the height of each of the small isosceles triangles at the top and bottom is $\frac{1}{2\sqrt{3}}$. Hence, the side length of the rhombus is

$$\sqrt{3} - \tfrac{1}{\sqrt{3}} = \tfrac{2\sqrt{3}}{3}.$$

The area is then

$$\begin{aligned} A_{\text{shaded}} &= 2 \cdot \tfrac{\sqrt{3}}{4} \cdot \left(\tfrac{2\sqrt{3}}{3}\right)^2 \\ &= \tfrac{2\sqrt{3}}{3}, \text{ as required.} \end{aligned}$$

847. We may assume, without loss of generality, that **a** and **b** are the standard unit vectors **i** and **j**. This is equivalent to rotating/reflecting the problem to align **a** and **b** with x and y axes.

The gradient of $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$ is $4/3$, while the gradient of $-4/5\mathbf{i} + \frac{3}{5}\mathbf{j}$ is $-3/4$. Since $-4/3 \times 3/4 = -1$, the vectors are perpendicular. And, since $(3/5, 4/5, 1)$ is a Pythagorean triad, we also know that these are unit vectors.

— Nota Bene —

Assuming something “without loss of generality” means assuming some specific value or case, while reassuring the reader that this assumption doesn’t affect the problem. Broadly, it shows that the case being assumed, while more specific than the given problem, may nevertheless be taken to *represent* the entire problem.

848. Splitting the fraction up,

$$\begin{aligned} \int_1^k 1 + \frac{1}{x^2}\,dx &= 4.8 \\ \Longrightarrow \left[x - \tfrac{1}{x}\right]_1^k &= 4.8 \\ \Longrightarrow \left(k - \tfrac{1}{k}\right) - \left(1 - 1\right) &= 4.8. \end{aligned}$$

Multiplying by k and simplifying,

$$\begin{aligned} & 5k^2 - 24k - 5 = 0 \\ \Longrightarrow\; & (5k + 1)(k - 5) = 0 \\ \Longrightarrow\; & k = -\tfrac{1}{5}, 5. \end{aligned}$$

849. The angle of projection which attains maximum range over flat ground is 45°. At this angle, the initial vertical velocity is $u \sin 45° = \frac{u}{\sqrt{2}}$. At the highest point, the vertical velocity is zero. The maximum height h is given by $v^2 = u^2 + 2as$:

$$\begin{aligned} 0 &= \left(\frac{u}{\sqrt{2}}\right)^2 - 2gh \\ \Longrightarrow h &= \frac{u^2}{4g}, \text{ as required.} \end{aligned}$$

850. The shortest path from a point to a line is along the normal. So, the line OP has gradient $-\frac{1}{2}$ and passes through $(0,0)$. Its equation is therefore $y=-\frac{1}{2}x$. Solving simultaneously with $y=2x+5$ gives $(-2,1)$.

851. (a) This is true, since 2^x+4 is never zero.
(b) This is true; both factors are zero at $x=0$.
(c) This is false; $x=\log_2 3$ is a counterexample.

852. This is a quartic in $\sin x$. Factorising,

$$\begin{aligned}&\sin^4 x+\sin^2 x=0\\ \Longrightarrow\ &\sin^2 x(\sin^2 x+1)=0.\end{aligned}$$

The latter factor has no real roots, so we require $\sin x=0$, which gives $x=0,180°$.

853. (a) This isn't well-defined. Over the domain $[0,1]$, the expression $2x$ has range $[0,2]$. There are values in $[0,2]$, for instance 2, which are not in the domain of g.
(b) This is well-defined. It has domain $[0,1]$ and range $[0,2]$.

854. The equations of a pair of parallel lines $ax+by=c$ and $ax+by=d$ form a counterexample. If $c=d$, then there are infinitely many solutions; if $c\neq d$, there are none.

855. Using $\cos\frac{\pi}{3}=\frac{1}{2}$ and $\cos 0=1$,

$$\begin{aligned}&\int_0^{\frac{\pi}{3}}\sin x\,dx\\ &=\Big[-\cos x\Big]_0^{\frac{\pi}{3}}\\ &=-\cos\tfrac{\pi}{3}+\cos 0\\ &=-\tfrac{1}{2}+1\\ &=\tfrac{1}{2}.\end{aligned}$$

856. Multiplying by x^2+y^2,

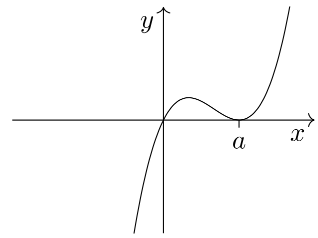
$$\begin{aligned}&x+y=x^2+y^2\\ \Longrightarrow\ &x^2-x+y^2-y=0\\ \Longrightarrow\ &\left(x-\tfrac{1}{2}\right)^2+\left(y-\tfrac{1}{2}\right)^2=\tfrac{1}{2}.\end{aligned}$$

The above is a circle, centre $(1/2,1/2)$, radius $\sqrt{2}/2$. However, $(0,0)$, which lies on this circle, does not satisfy the original equation, since $0/0$ is undefined. Hence, the locus of points satisfying the original equation is a circle minus a point. QED.

857. The original possibility space is $\{1,2,3,4,5,6\}$, giving probability $1/2$. With the information "The score is even", the space is restricted to $\{2,4,6\}$, of which $2/3$ are at least 4. Hence, the information increases the relevant probability from $1/2$ to $2/3$.

858. Differentiating, we get $\frac{dy}{dx}=x$. Substituting $x=a$, the gradient of the tangent is a. By definition, the gradient of a straight line is $\tan\theta$, where θ is its angle of inclination. So, $\tan\theta=a$, as required.

859. (a) The curve crosses the x axis at 0 (single root) and touches it at $x=a$ (double root):

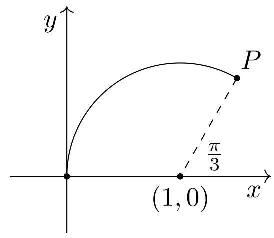


(b) The limits are $x=0$ to $x=a$. So, the area enclosed by the graph and the x axis is

$$\begin{aligned}&\int_0^a 12x(x-a)^2\,dx\\ &=\int_0^a 12x^3-24ax^2+12a^2x\,dx\\ &=\Big[3x^4-8ax^3+6a^2x^2\Big]_0^a\\ &=(3a^4-8a^4+6a^4)-(0)\\ &=a^4,\ \text{as required.}\end{aligned}$$

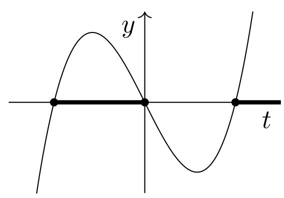
860. (a) $(a,c)\cap[b,d]=[b,c)$,
(b) $(a,c]\cup[b,d)=(a,d)$,
(c) $(a,b]\cap[c,d)=\varnothing$.

861. The vector from $(1,0)$ to P is $\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}$. This is directed at angle $\frac{\pi}{3}$ radians and has length 1. So, P moves along the arc of a circle of radius 1:



The image under this rotation is the origin.

862. The boundary equation $t^3-4t=0$ has roots $t=-2,0,2$. And $y=t^3-4t$ is a positive cubic:



The solution is consists of all x values for which the y value is non-negative. This gives the solution

$$t\in[-2,0]\cup[2,\infty).$$

863. Putting the fractions over a common denominator,

$$\begin{aligned} y &= \frac{1}{1+x} + \frac{1}{1-x} \\ \implies y &= \frac{1}{1-x^2} \\ \implies 1 - x^2 &= \tfrac{1}{y} \\ \implies x &= \pm\sqrt{1 - \tfrac{1}{y}}. \end{aligned}$$

864. This is equivalent to saying that $(x+1)$ is not a factor of the cubic. So, we need only evaluate the cubic at $x = -1$. This gives -6. Hence, by the factor theorem, $(x+1)$ is not a factor of the cubic, and the ratio cannot be rearranged to the form suggested.

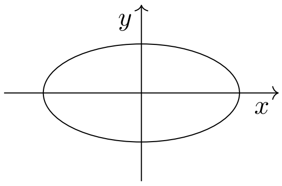
865. (a) The smallest angle is

$$\begin{aligned} &180° - \arccos\tfrac{1}{8} - \arccos\tfrac{9}{16} \\ &= 41.4096...° \\ &= 41.4° \text{ (1dp)}. \end{aligned}$$

(b) The shortest side, length 4 cm, is opposite the smallest angle 41.4°. So, by the sine rule, the other lengths are

$$\begin{aligned} \frac{4\sin(\arccos\frac{9}{16})}{\sin 41.4°} &= 5, \\ \frac{4\sin(\arccos\frac{1}{8})}{\sin 41.4°} &= 6. \end{aligned}$$

866. This is true. The curve is an ellipse, centred on the origin. A circle centred on the origin is normal to the coordinate axes, and the stretch factor $\frac{1}{2}$ in the y direction does not change this fact.



867. If $\mathbf{F} = p\mathbf{G}$ for some $p \in \mathbb{R}$, then the forces $\mathbf{F}$ and $\mathbf{G}$ are parallel. The third force cannot then have a component perpendicular to these. So, it must also be parallel to $\mathbf{G}$ (and $\mathbf{F}$). This is expressed in the statement "$\mathbf{H} = q\mathbf{G}$ for some $q \in \mathbb{R}$."

— Alternative Method —

If the object is in equilibrium under the action of three forces, then $\mathbf{F} + \mathbf{G} + \mathbf{H} = 0$. Substituting in $\mathbf{F} = p\mathbf{G}$,

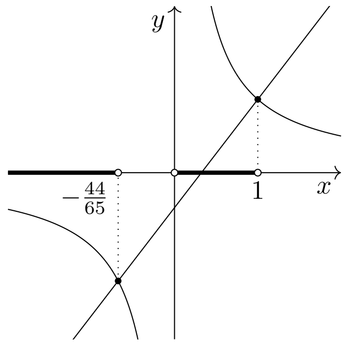
$$\begin{aligned} p\mathbf{G} + \mathbf{G} + \mathbf{H} &= 0 \\ \implies \mathbf{H} &= (-1-p)\mathbf{G}. \end{aligned}$$

Defining $q = -1 - p$, this is the required result.

868. Setting up the boundary equation,

$$\begin{aligned} 1.30x - 0.42 &= \frac{0.88}{x} \\ \implies 65x^2 - 21x - 44 &= 0 \\ \implies (65x+44)(x-1) &= 0 \\ \implies x &= 1, -\tfrac{44}{65}. \end{aligned}$$

Sketching the LHS and RHS



The solution to the inequality is all x values for which the hyperbola is above the straight line. This is $x \in (-\infty, -44/65) \cup (0, 1)$.

869. The information given is a quadratic equation in Δ. Factorising, we get $\Delta(\Delta - 1) = 0$, so $\Delta = 0$ or $\Delta = 1$. The discriminant is either zero or positive, so $n = 1$ or $n = 2$.

870. In both (a) and (b), the ratio of the magnitudes of numerator and denominator approaches 1. The only difference is the sign of the numerator. This gives

(a) $\displaystyle\lim_{x\to+\infty} \frac{x}{|x|+1} = +1,$

(b) $\displaystyle\lim_{x\to-\infty} \frac{x}{|x|+1} = -1.$

871. Since $(9, 3)$ is in the positive quadrant, we can use $y = \sqrt{x}$. Differentiating,

$$\frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}}.$$

The gradient of the tangent at $(9, 3)$ is $\frac{1}{6}$, so the normal gradient is -6. The equation of the normal is therefore

$$y - 3 = -6(x - 9).$$

Setting $y = 0$, we get $x = \frac{19}{2}$, as required.

— Alternative Method —

Differentiating $x = y^2$ with respect to y,

$$\frac{dx}{dy} = 2y.$$

At $(9, 3)$, dx/dy has value 6. So, the gradient of the tangent is $\frac{1}{6}$ and the gradient of the normal is -6. The equation of the normal is therefore $y - 3 = -6(x - 9)$. At $y = 0$, $x = \frac{19}{2}$.

872. (a) The relevant constant acceleration formula is $v^2 = u^2 + 2as$. Since the initial velocity u is squared, it makes no difference whether it is positive or negative. So, a projectile will land with the same velocity whether it launched vertically upwards or vertically downwards.

(b) Yes. The speeds for both would be reduced, but the reduction for the up projectile would be greater. Without any air resistance, the up projectile would return to its initial height with its initial speed, becoming identical to the down projectile; with air resistance, however, it will have lost some speed.

873. $\triangle AXC$ is right-angled, congruent to $\triangle ABC$. Its perpendicular sides have length 1, so its area is $\frac{1}{2}$.

874. Differentiating twice,

$$\begin{aligned} & y = x^4 \\ \Longrightarrow\ & \frac{dy}{dx} = 4x^3 \\ \Longrightarrow\ & \frac{d^2y}{dx^2} = 12x^2. \end{aligned}$$

We can rewrite this as

$$\frac{d^2y}{dx^2} = 12x^2 = 12\sqrt{x^4} = 12\sqrt{y}.$$

So, $y = x^4$ satisfies the DE.

875. (a) The probabilities are

i. $1/5$,
ii. $1/3$,
iii. $1/5$,
iv. $\dfrac{k+1}{5k+1}$.

(b) Dividing top and bottom by k, the fractions $\frac{1}{k}$ tend to zero:

$$\lim_{k\to\infty} \frac{k+1}{5k+1} = \lim_{k\to\infty} \frac{1+\frac{1}{k}}{5+\frac{1}{k}} = \frac{1}{5}.$$

876. (a) Using the cosine rule, the squared lengths of the diagonals of the rhombus are given by

$$\begin{aligned} d_1^2 &= l^2\big(2 - 2\cos\tfrac{\pi}{4}\big), \\ d_2^2 &= l^2\big(2 - 2\cos\tfrac{3\pi}{4}\big). \end{aligned}$$

Using standard trig values, these simplify to

$$\begin{aligned} d_1^2 &= l^2\big(2 - \sqrt{2}\big), \\ d_2^2 &= l^2\big(2 + \sqrt{2}\big). \end{aligned}$$

Taking the positive square root of each,

$$\begin{aligned} d_1 &= l\sqrt{2-\sqrt{2}}, \\ d_2 &= l\sqrt{2+\sqrt{2}}. \end{aligned}$$

(b) Counting right-angled triangles, the area of the rhombus is given by $\frac{1}{2}d_1d_2$. Using the values from part (a),

$$\begin{aligned} A &= \tfrac{1}{2}l^2\sqrt{2-\sqrt{2}}\sqrt{2+\sqrt{2}} \\ &= \tfrac{1}{2}l^2\sqrt{4-2} \\ &= \tfrac{\sqrt{2}}{2}l^2, \text{ as required.} \end{aligned}$$

877. The discriminants of each quadratic factor are $\Delta_1 = 0$ and $\Delta_2 = -11$, which is negative. These give 1 and 0 real roots respectively. Hence, the quartic has exactly 1 real root.

878. These are a pair of linear equations in x^2 and y^2. Solving simultaneously gives $x^2 = 9$ and $y^2 = 1$. The signs of x and y do not affect each other. So, there are four solutions $(\pm 3, \pm 1)$, where the $\pm$ signs are independent.

879. (a) Since f and g commute, we know that

$$(kx)^2 \equiv kx^2.$$

This requires $k^2 = k$, which has roots $k = 0, 1$.

(b) This follows directly from an index law:

$$\mathrm{f}\big(\mathrm{g}(x)\big) = \big(x^b\big)^a \equiv x^{ab} \equiv \big(x^a\big)^b = \mathrm{g}\big(\mathrm{f}(x)\big).$$

So, $\mathrm{f}(x) = x^a$ and $\mathrm{g}(x) = x^b$ always commute.

880. We know that $l = r\theta$ and $A = \frac{1}{2}r^2\theta$, where r is the radius of the sector. Substituting the former into the latter,

$$\begin{aligned} A &= \tfrac{1}{2}\left(\frac{l}{\theta}\right)^2\theta \\ &\equiv \frac{l^2}{2\theta}, \text{ as required.} \end{aligned}$$

881. There is a common factor of $(x-2)^3$. It is much easier to take this factor out first:

$$\begin{aligned} & (x-2)^4 - (x-2)^3 = 0 \\ \Longrightarrow\ & (x-2)^3(x-2-1) = 0 \\ \Longrightarrow\ & (x-2)^3(x-3) = 0 \\ \Longrightarrow\ & x = 2, 3. \end{aligned}$$

882. Since $\cos 36^\circ = \frac{1}{4}\big(1+\sqrt{5}\big)$, we know that

$$\begin{aligned} \sec\theta &= \frac{4}{1+\sqrt{5}} \\ &= \frac{4\big(1-\sqrt{5}\big)}{1-5} \\ &= \sqrt{5} - 1. \end{aligned}$$

883. To find a root and thereby a factor, we set up a Newton-Raphson iteration:

$$x_{n+1} = x_n - \frac{x^3 + 4x^2 - 27x - 90}{3x^2 + 8x - 27}.$$

Running this with $x_0 = 0$ yields $x_n \to -3$. So $x = -3$ is a root, and $(x + 3)$ is a factor. This gives

$$\begin{aligned} & x^3 + 4x^2 - 27x - 90 = 0 \\ \Longrightarrow\ & (x+3)(x^2 + x - 30) = 0 \\ \Longrightarrow\ & (x+3)(x+6)(x-5) = 0 \\ \Longrightarrow\ & x = -6, -3, 5. \end{aligned}$$

— Nota Bene —

It would be reasonable here to find the requisite root simply using a calculator table facility. The roots are all integers, so you can find them easily enough. I have used the N-R method above just to show that the process doesn't have to be a lottery. You can always find the roots if you need them.

On a separate note, the long division for taking out the factor of $(x + 3)$ is

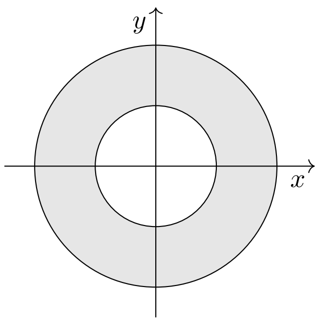
$$\begin{array}{r l} & x^2 \quad + x - 30 \\ x+3 \,\big) & x^3 + 4x^2 - 27x - 90 \\ & -\, x^3 - 3x^2 \\ & \quad x^2 - 27x \\ & \quad -\, x^2 \ - 3x \\ & \qquad -\, 30x - 90 \\ & \qquad\ \ 30x + 90 \\ & \qquad\qquad 0 \end{array}$$

It's better to shortcut this method.

884. (a) The boundary equation is solved when one of the factors is zero, i.e. $x^2 + y^2 = 1, 4$. These are two circles, centred on the origin, with radii 1 and 2.

For the original LHS to be negative, exactly one of its factors must be negative. So the region is the annulus $1 \leq x^2 + y^2 \leq 4$ between the circles.

(b) Sketch of $1 \leq x^2 + y^2 \leq 4$



885. The acceleration under gravity on a smooth slope of angle θ is $g \sin\theta$. In this case, $a = \frac{\sqrt{2}}{2} g$. Using $s = ut + \frac{1}{2}at^2$,

$$\begin{aligned} & 1 = \tfrac{\sqrt{2}}{4} g t^2 \\ \Longrightarrow\ & t = \pm 0.537229... \end{aligned}$$

So, $t = 0.537$ s (3sf).

886. Putting the fractions over a common denominator,

$$\begin{aligned} & \frac{\sqrt{2}+1}{\sqrt{2}-1} + \frac{\sqrt{2}-1}{\sqrt{2}+1} \\ = & \frac{(\sqrt{2}+1)^2 + (\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)} \\ = & \frac{(2 + 2\sqrt{2} + 1) + (2 - 2\sqrt{2} + 1)}{1} \\ = & \ 6. \end{aligned}$$

887. (a) A general line through $(0, -4)$ is $y = mx - 4$.

(b) Intersections are given by $x^2 + 2x = mx - 4$, which simplifies to $x^2 + (2 - m)x + 4 = 0$.

(c) For a tangent, we require the quadratic in (b) to have exactly one root. So, the discriminant must be zero:

$$\Delta = (2 - m)^2 - 16 = 0.$$

Solving for m gives $m = -2, 6$.

(d) The tangent lines have equations $y = -2x - 4$ and $y = 6x - 4$. Solving each simultaneously with $y = x^2 + 2x$ gives the possible coordinates of P as $(-2, 0)$ or $(2, 8)$.

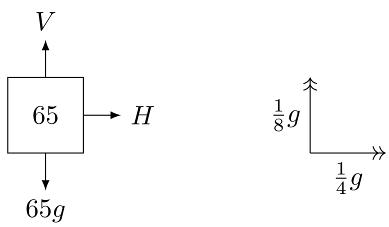
888. The sum of two rational numbers can be written in terms of integers a, b, c, d as

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

Since a, b, c, d are integers, we know that $ad + bc$ and bd are integers. So, the sum can be expressed as a quotient of integers, and is rational. □

889. The implication is $\Longleftarrow$, from the second to the first. The first equation has a root $x = -1$, like the second, but also $x = 0$, unlike the second. So, $x = 0$ is a counterexample to both $\Longrightarrow$ and $\Longleftrightarrow$.

890. The airliner, via the seat and the floor, exerts a contact force on the passenger. We model this in its horizontal and vertical components, which have magnitudes H and V:



The equations of motion are $H = 65 \cdot \frac{1}{4}g$ and $V - 65g = 65 \cdot \frac{1}{8}g$. Evaluating these, $H = 159.25$ and $V = 716.625$. Using Pythagoras,

$$\begin{aligned} C &= \sqrt{H^2 + V^2} \\ &= \sqrt{159.25^2 + 716.625^2} \\ &= 734.106... \end{aligned}$$

So, the contact force has magnitude 734 N (3sf).

891. The possibility space is:

	1	2	3	4	5	6
1						✓
2						✓
3						✓
4						✓
5						✓
6	✓	✓	✓	✓	✓	✓

Counting outcomes gives $p = \frac{11}{36}$.

892. (a) Taking out $(x-2)^2$ for the double root:

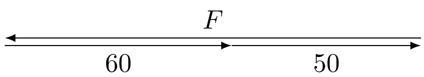
$$\begin{aligned} & x^3 + x^2 - 16x - 20 = 0 \\ \Longrightarrow\ & (x-2)^2(x+5) = 0 \\ \Longrightarrow\ & x = 2, -5. \end{aligned}$$

(b) Differentiating, $f'(x) = 3x^2 + 2x - 16$.

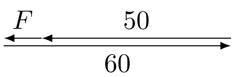
(c) $f'(2) = 0$, $f'(-5) = -105$.

(d) Since $f(x) = 0$ has a double root at $x = 2$, the root is also a turning point, hence $f'(2) = 0$. However, $x = -5$ is a single root, so the curve $y = f(x)$ crosses the x axis with $f'(-5) \neq 0$.

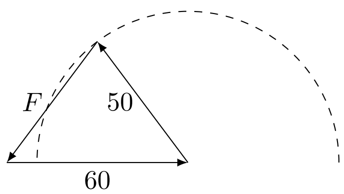
893. The magnitude of F is greatest if the other two forces are parallel:



And least if the other two forces are antiparallel:



All magnitudes between these bounds are possible, as shown below:



Hence, the set of possible values is $[10, 110]$ N.

894. If f' is a linear function, then $f'(x) = px + q$, for some constants p and q. Integrating,

$$\begin{aligned} f(x) &= \int px + q\, dx \\ &= \tfrac{1}{2}px^2 + qx + c. \end{aligned}$$

Renaming the constants, $f(x) = ax^2 + bx + c$. This is all quadratic ($a \neq 0$) or linear ($a = 0$) functions.

895. The straight sections have length $6r$. The curved sections form one full circumference of length $2\pi r$. So, the total length is $2r(\pi + 3)$.

896. The factorials are $1, 2, 6, 24, ...$ Hence, $n = 4$ is a counterexample to the student's claim, since $n! + 1 = 25$, which is non-prime.

897. Multiplying out, the integrand is $x^n(1-x^2)$, which simplifies to $x^n - x^{n+2}$. Carrying out the definite integral,

$$\begin{aligned} & \int_0^1 x^n - x^{n+2}\, dx \\ \equiv\ & \left[\tfrac{1}{n+1}x^{n+1} - \tfrac{1}{n+3}x^{n+3}\right]_0^1 \\ \equiv\ & \frac{1}{n+1} - \frac{1}{n+3}. \end{aligned}$$

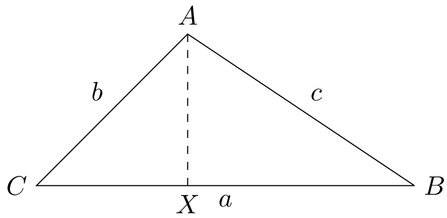
So, the original equation is

$$\frac{1}{n+1} - \frac{1}{n+3} = \frac{2}{35}$$

Multiplying by $35(n+1)(n+3)$,

$$\begin{aligned} & 35(n+3) - 35(n+1) = 2(n+1)(n+3) \\ \Longrightarrow\ & 35n + 105 - 35n - 35 = 2n^2 + 8n + 6 \\ \Longrightarrow\ & n^2 + 4n - 32 = 0 \\ \Longrightarrow\ & (n+8)(n-4) = 0 \\ \Longrightarrow\ & n = -8, 4. \end{aligned}$$

898. We set up triangle ABC as shown below, dropping a perpendicular from the vertex A and labelling its endpoint X:



Using right-angled trigonometry, $|CX| = b\cos C$, $|AX| = b\sin C$, and $|XB| = a - b\cos C$. We then apply Pythagoras in triangle ABX:

$$\begin{aligned} & c^2 = (a - b\cos C)^2 + (b\cos C)^2 \\ \Longrightarrow\ & c^2 = a^2 - 2ab\cos C + b^2\cos^2 C + b^2\sin^2 C. \end{aligned}$$

By the first Pythagorean identity, the last two terms sum to b^2, which gives the cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

899. (a) Translating into algebra, we have $x_A = 0.2t^2$ and $x_B = 0.3(t-2)^2 - 10$. The positions are the same when

$$\begin{aligned} & 0.2t^2 = 0.3(t-2)^2 - 10 \\ \Longrightarrow\; & 2t^2 = 3t^2 - 12t + 12 - 100 \\ \Longrightarrow\; & t^2 - 12t - 88 = 0. \end{aligned}$$

(b) Solving for the positive root, $t = 6 + 2\sqrt{31}$. Substituting this back into either expression for position gives 58.7 m (3sf) from O. Car B starts 10 metres back, so it travels 68.7 m (3sf) before overtaking.

900. (a) Counting successful outcomes,

i. $\mathbb{P}(X \in B) = \frac{1}{2}$,

ii. $\mathbb{P}(X \in C) = \frac{1}{2}$,

iii. $\mathbb{P}(X \in B \cap C) = \frac{1}{5}$.

(b) $\mathbb{P}(X \in B) \times \mathbb{P}(X \in C) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. This is not equal to $\mathbb{P}(X \in B \cap C)$, which is $\frac{1}{5}$. So, the events $X \in B$ and $X \in C$ are not independent.

——— End of 9th Hundred ———